Algorithm Problem Solving (APS): Greedy Method

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Example: The Change Problem (USA Currency)

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  - **C** = {1¢ (penny), 5¢ (nickel), 10¢ (dime), 25¢ (quarter)}

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- **The issue: your problem formulation was not specific!**

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- A *correct* solution optimizing the *objective function* is **optimal**

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- You should be happy receiving any such solution
  - If not, you need to fix your objective function!

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- **C** = {1¢ (penny), 5¢ (nickel), 10¢ (dime), 25¢ (quarter)}
- Imagine I owe you 42¢. How should I choose the coins to give you?

Let's solve the problem!

Revisiting the Change Problem (USA Currency)

```
Algorithm change_USA(x,C):
    change ← empty list
    For each coin c in C (descending order):
        While x >= c:
            Add c to change
            x ← x - c
    Return change
```

Revisiting the Change Problem (USA Currency)

```
Algorithm change_USA(x,C):
    change ← empty list
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Does this work for *any arbitrary* currency?

```
            Add c to change
            x ← x - c
    Return change
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    - **Exhaustive:** Simply looking at every possible solution
- When we try to cleverly search for an optimal solution more quickly:
    - **Global:** We can look at entire solutions at a time
    - **Local:** We can break solutions into parts and optimize part-by-part

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- **Note that this does not always work!!!**
    - We often skip what's immediately best to improve in the long-run
    - Example: Buying vs. leasing a car
- Thus, it's important to prove the correctness of a Greedy Algorithm

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- Imagine I owe you 6¢. How should I choose the coins to give you?
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  - The optimal solution is [3¢, 3¢]
  - **Our greedy algorithm doesn't work for all possible currencies!!!**

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- Greedy: Take the best immediate benefit and ignore opportunity costs
    - Optimal when immediate benefit outweighs opportunity costs

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  - You charge a flat rate, regardless of the length of the event
  - Thus, you want to schedule as many events as possible
  - However, events cannot overlap

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- **Input:** All $\boldsymbol{n}$ possible events $\boldsymbol{E} = [(start_1, end_1), \ldots, (start_{\boldsymbol{n}}, end_{\boldsymbol{n}})]$

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- If we wanted to design a greedy algorithm, what would we optimize?

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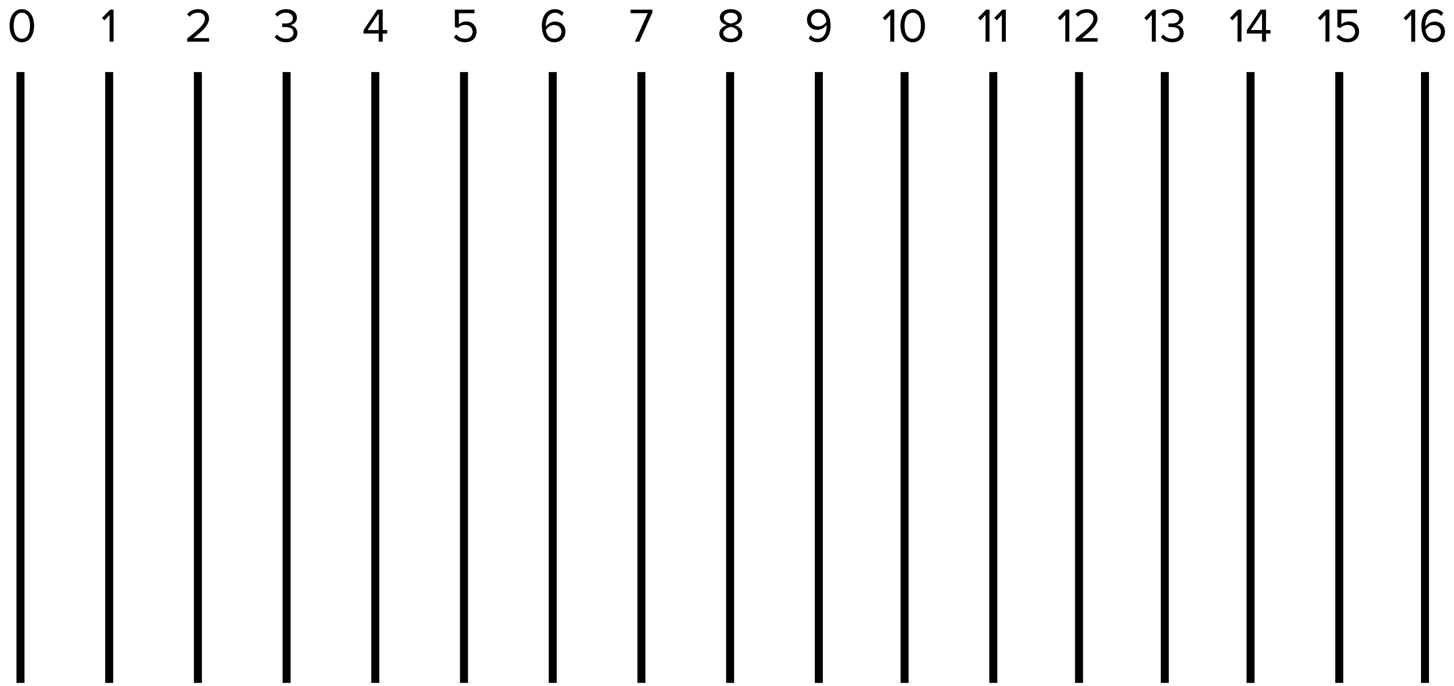
- **Input:** All $\boldsymbol{n}$ possible events $\boldsymbol{E}$ = [($start_1$, end_1), ..., ($start_{\boldsymbol{n}}$, $end_{\boldsymbol{n}}$)]
- **Output:** A non-overlapping subset of $\boldsymbol{E}$ maximizing its size
- If we wanted to design a greedy algorithm, what would we optimize?
  - Shortest duration?
  - Earliest start time?
  - Fewest conflicts?
  - Earliest end time?

Counterexample: Shortest Duration



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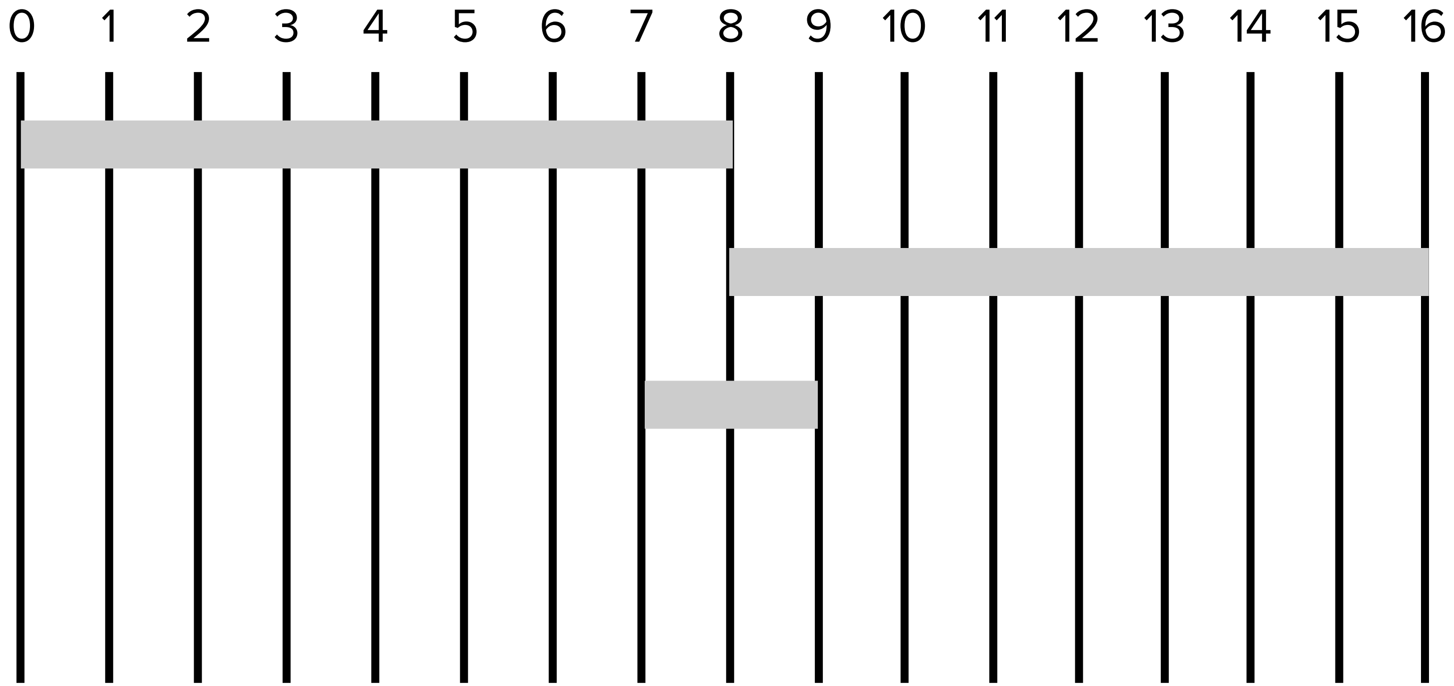

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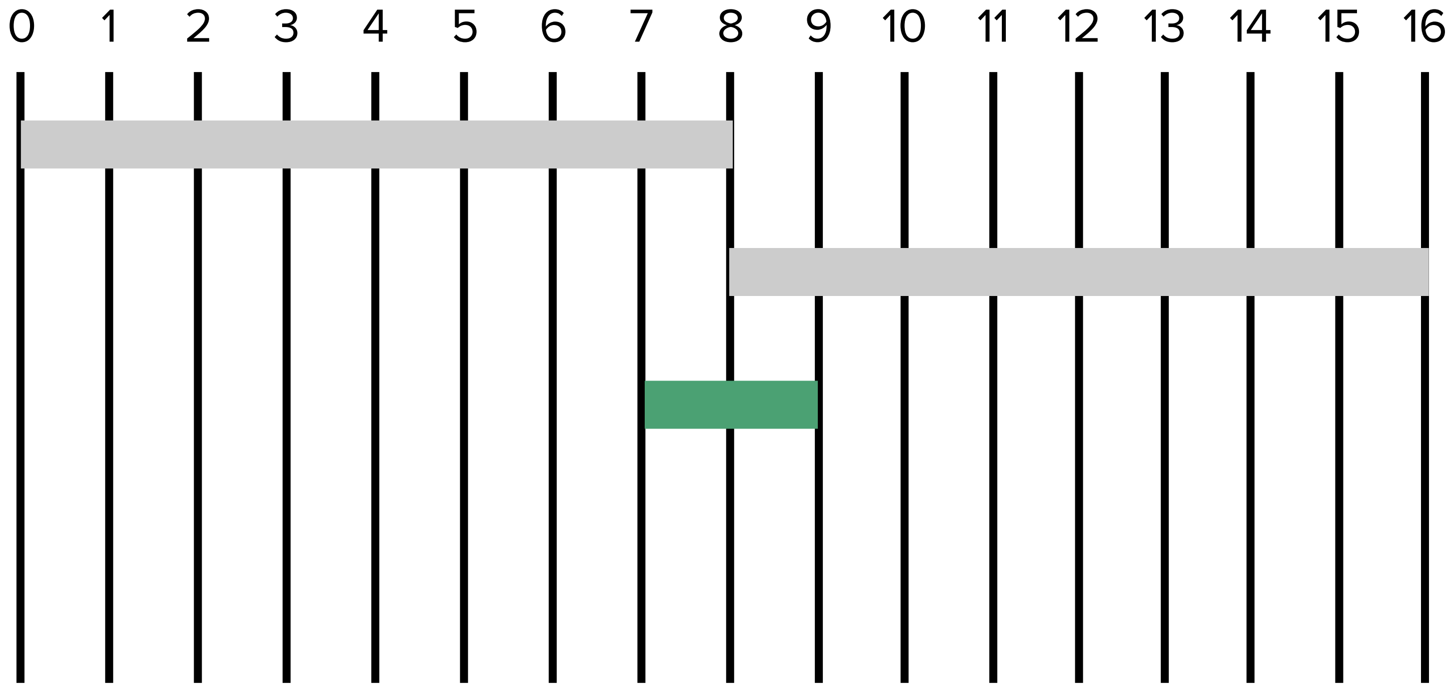

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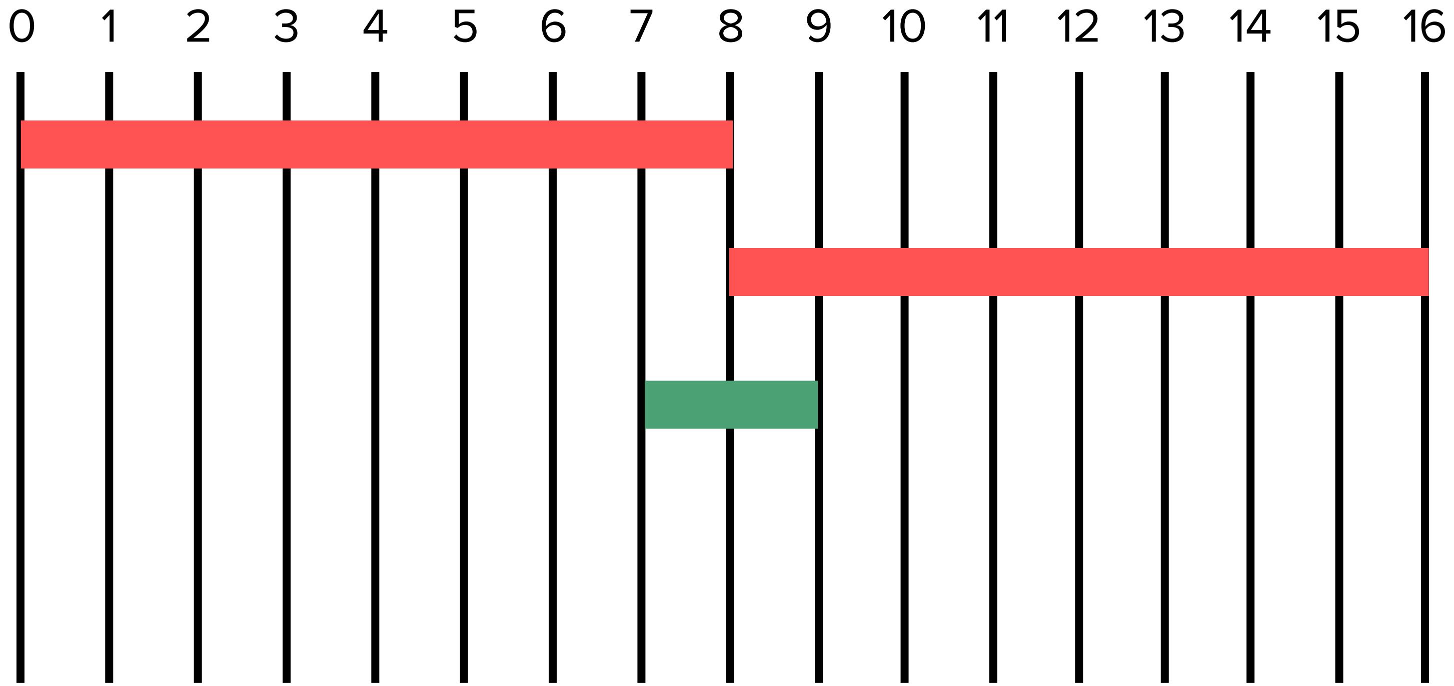

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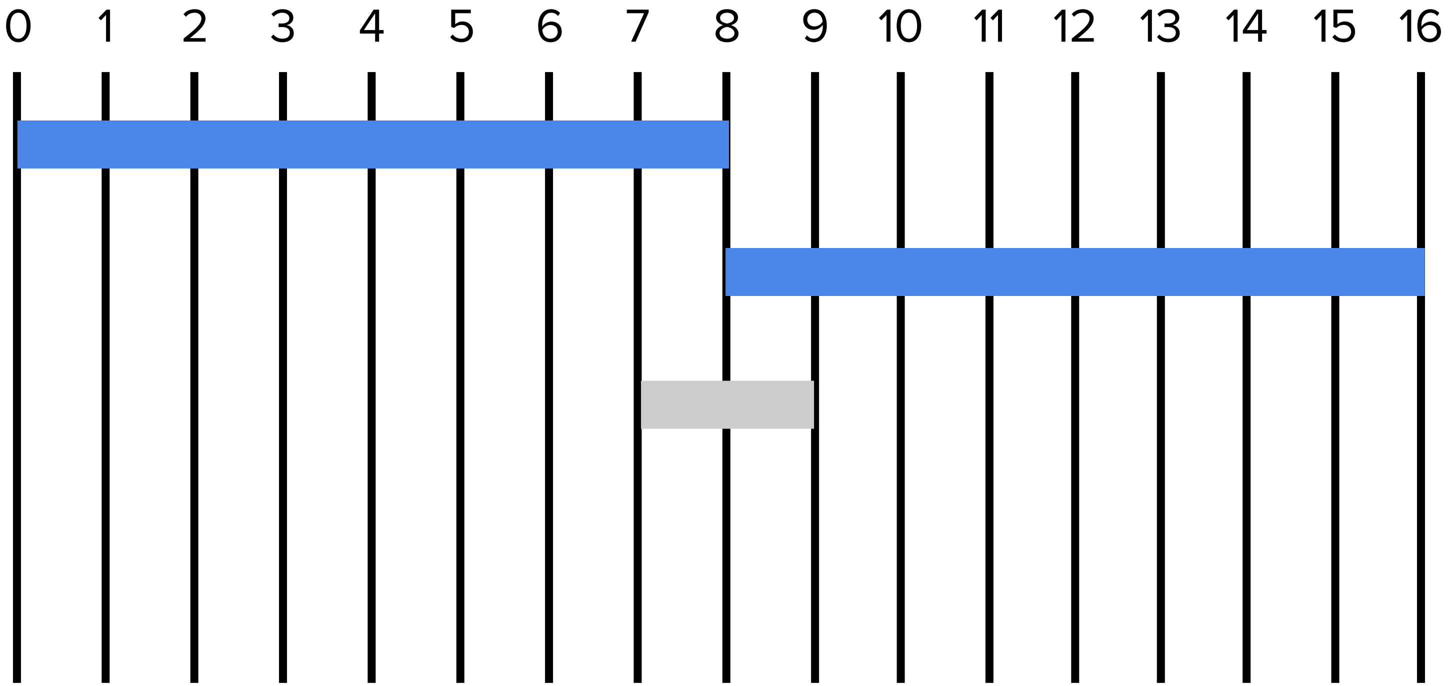

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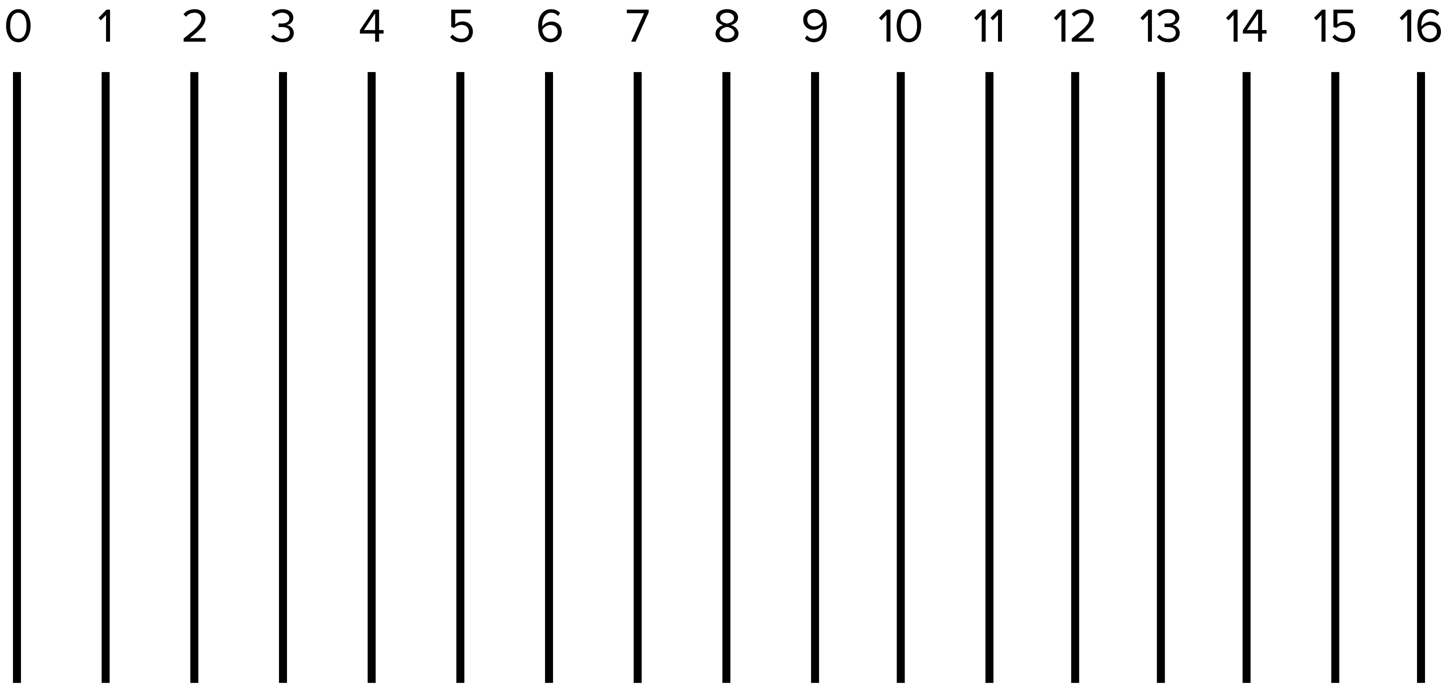

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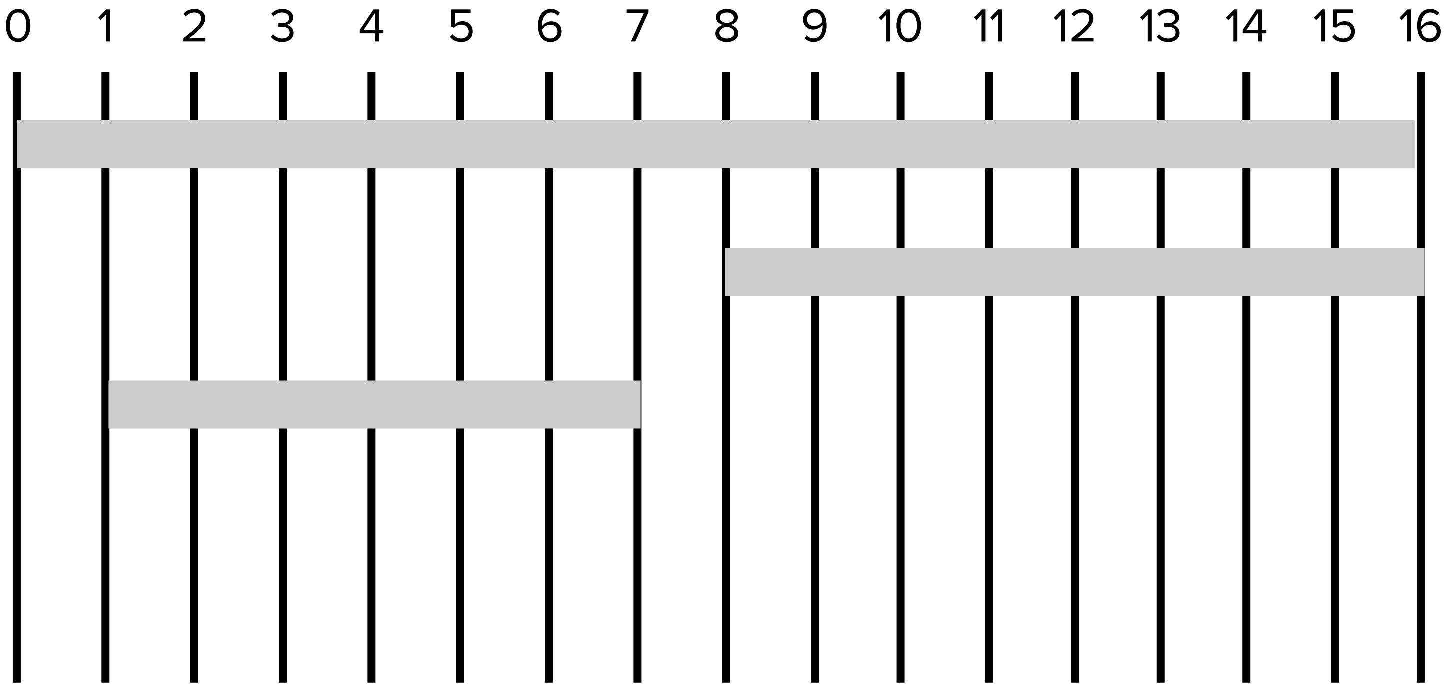

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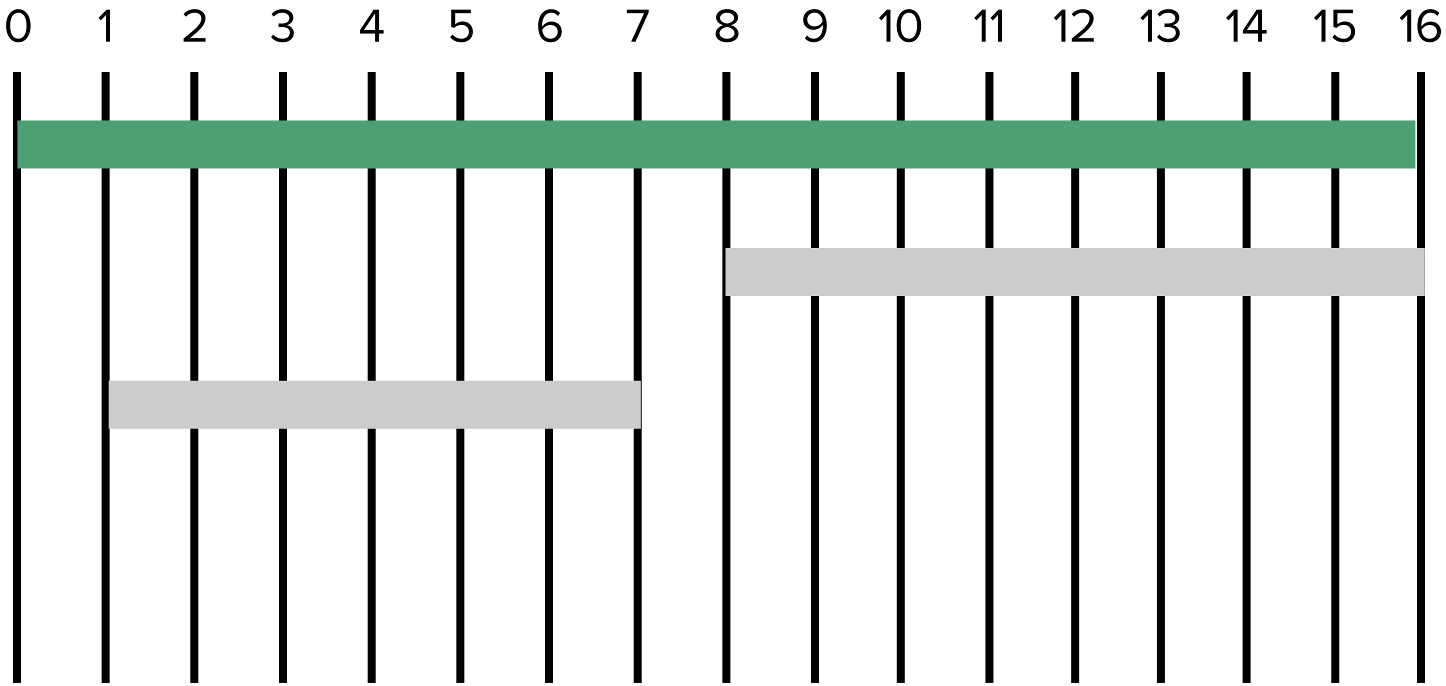

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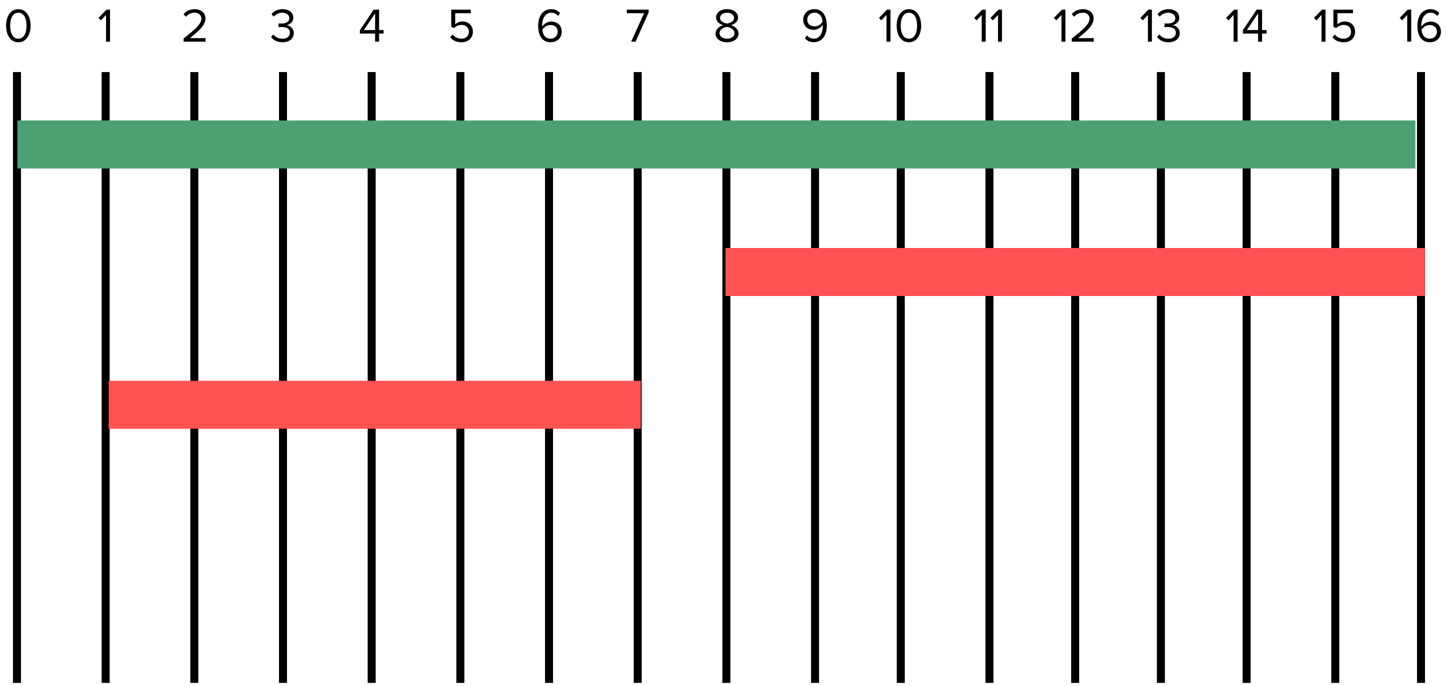

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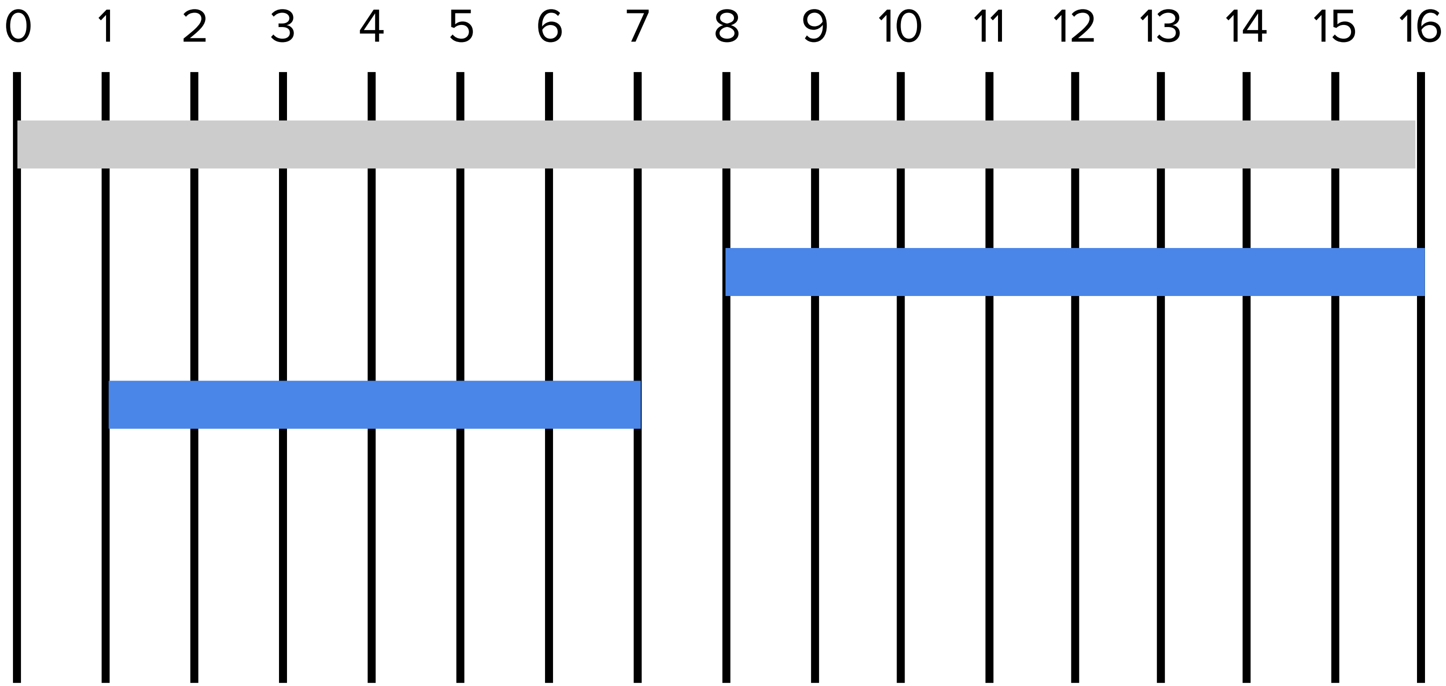

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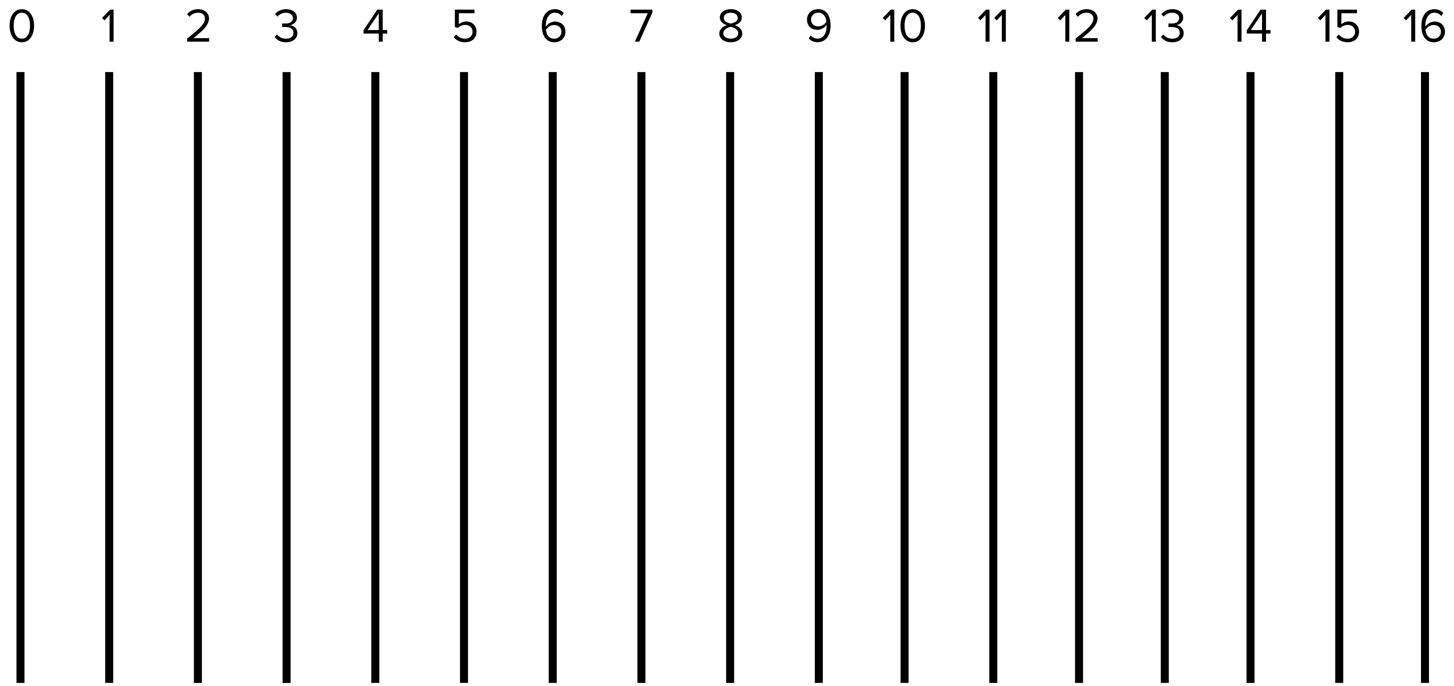

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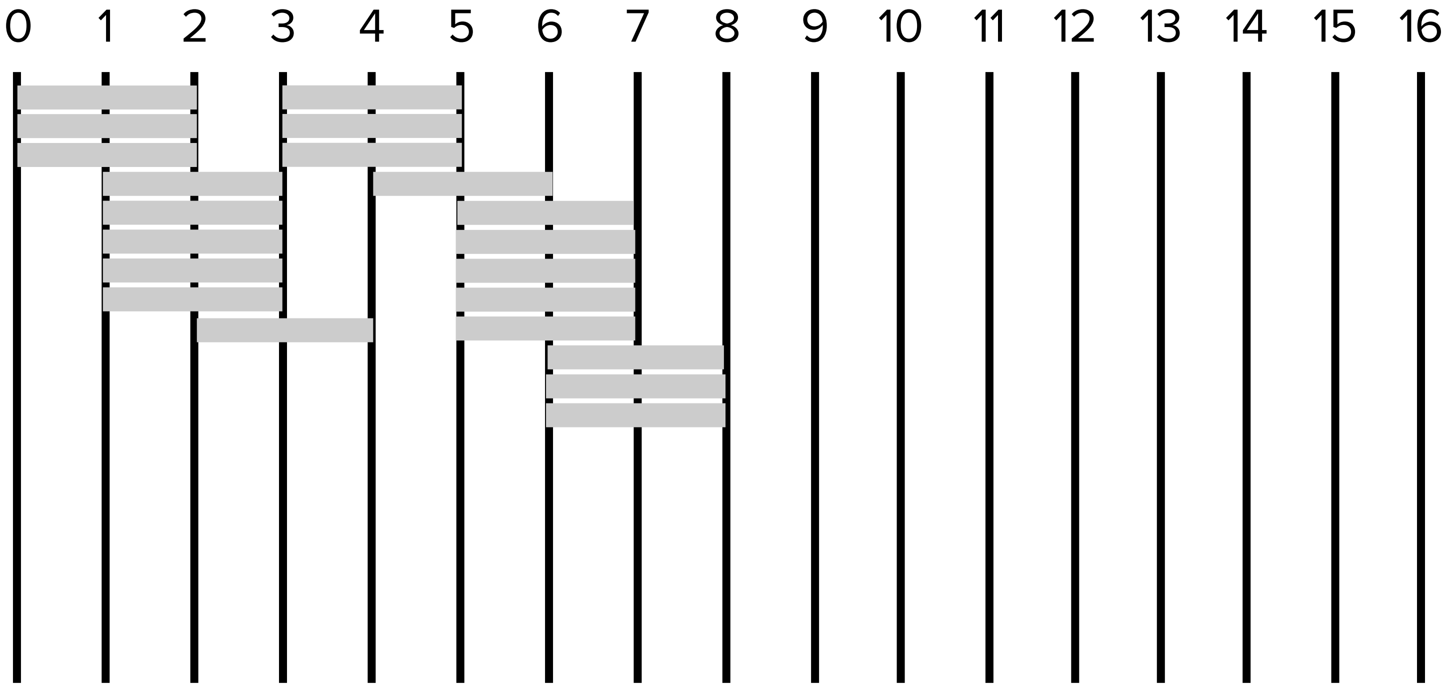

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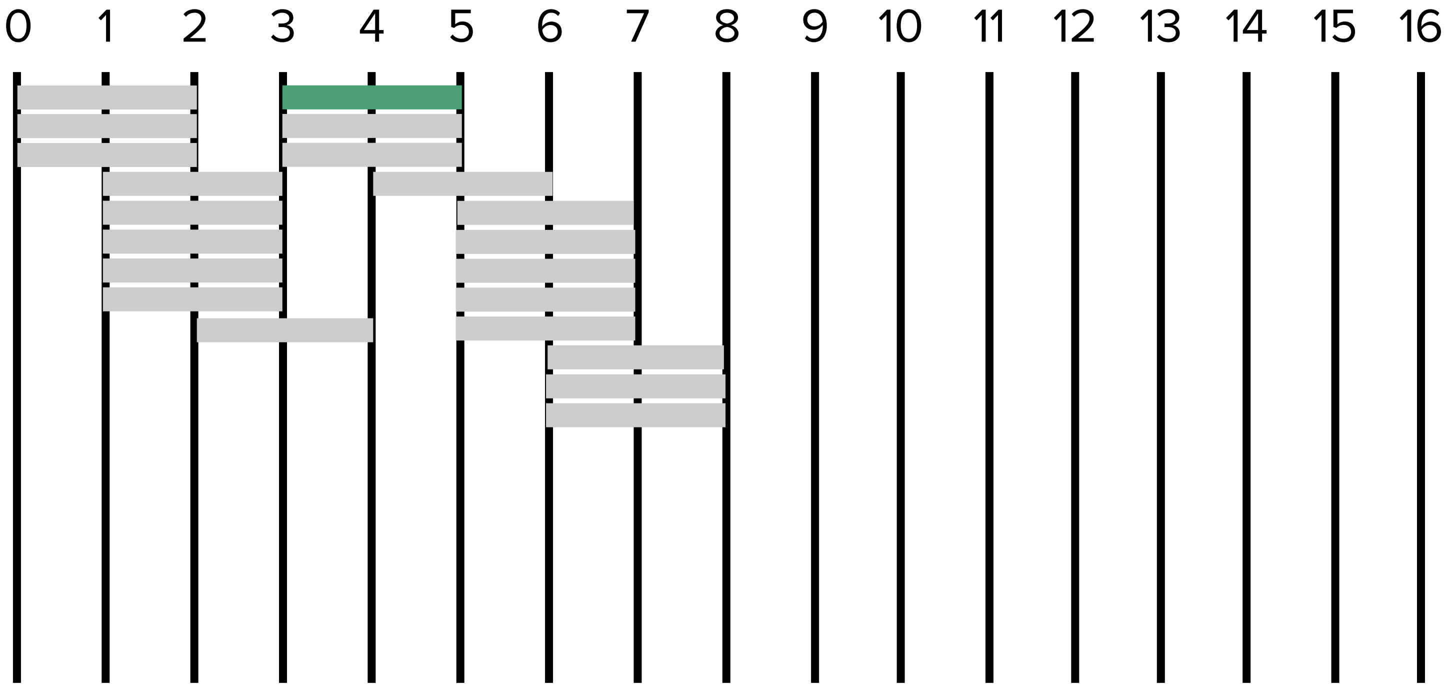

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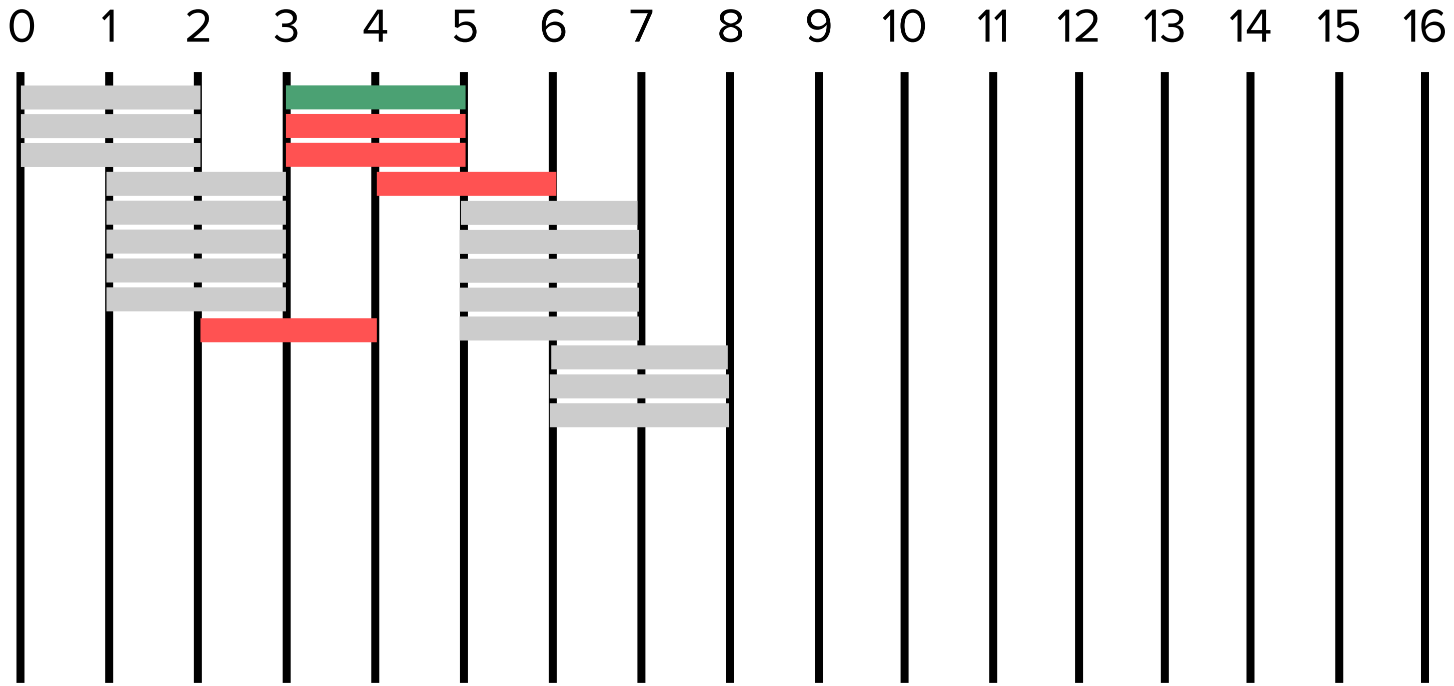

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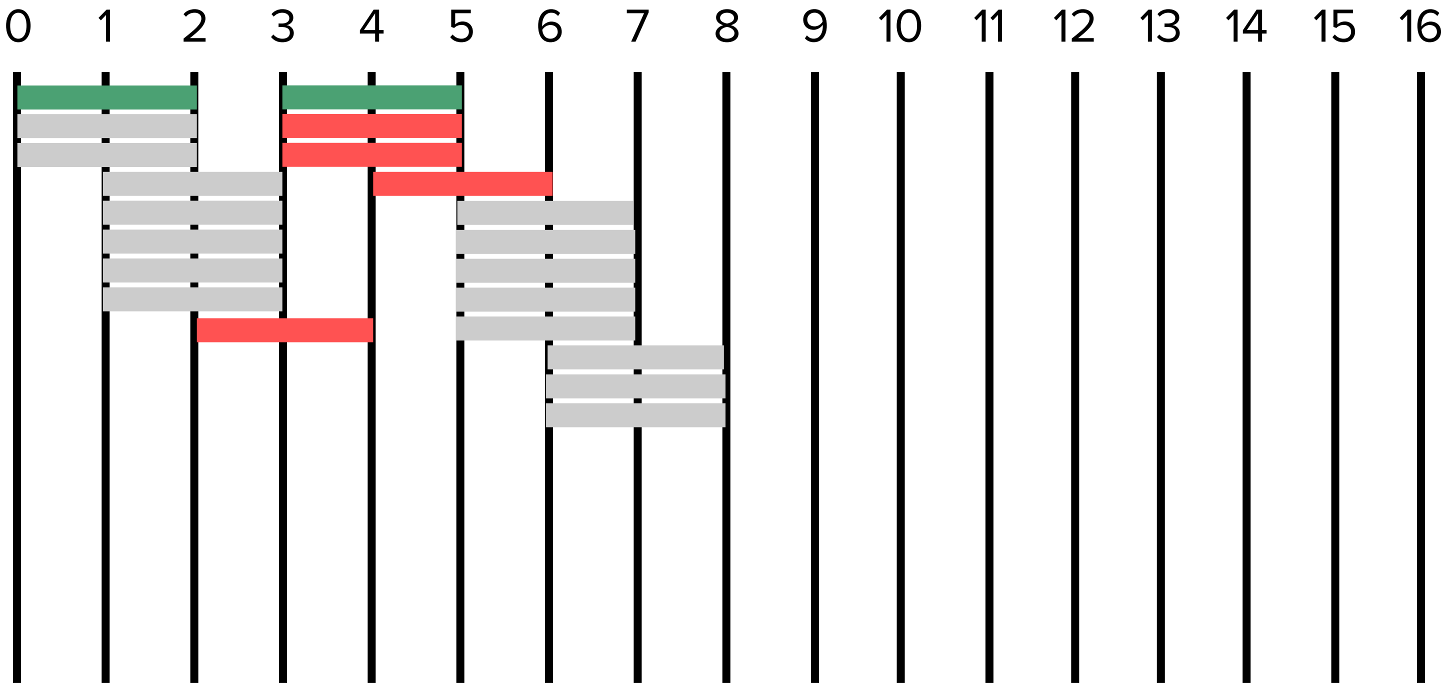

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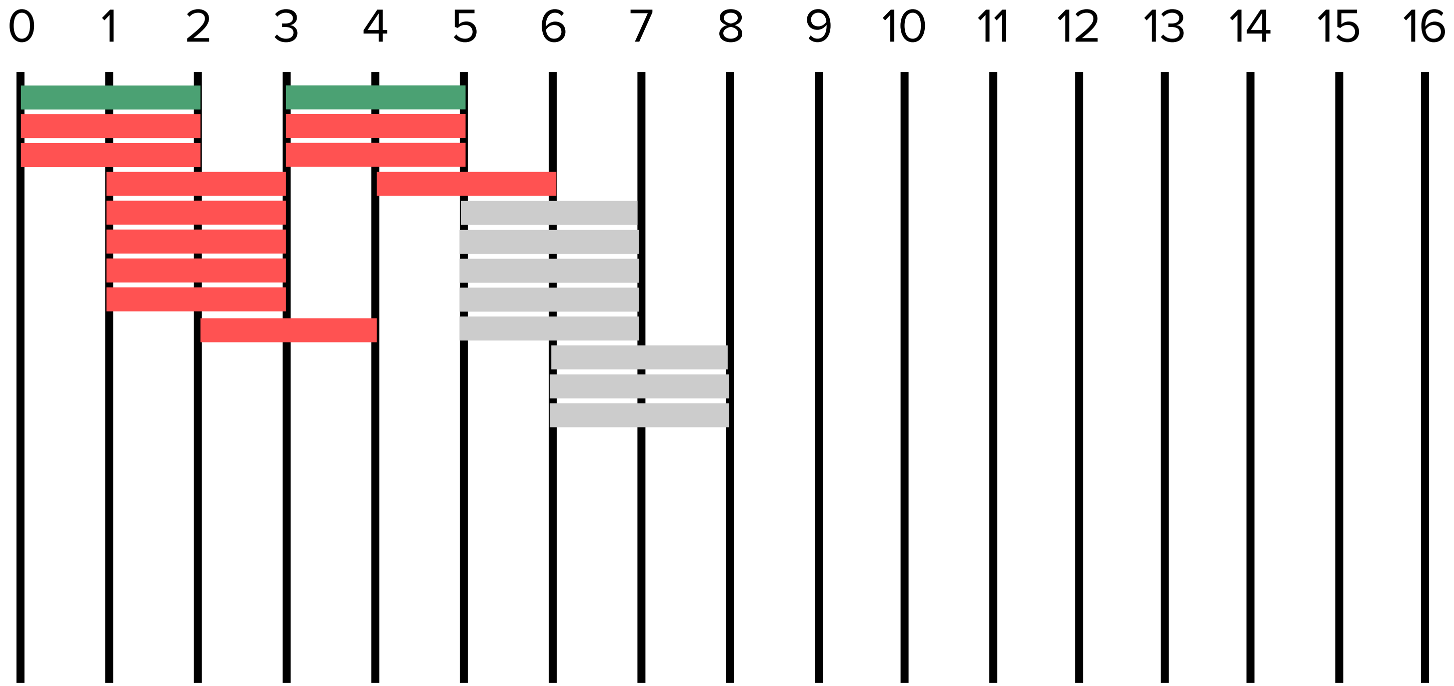

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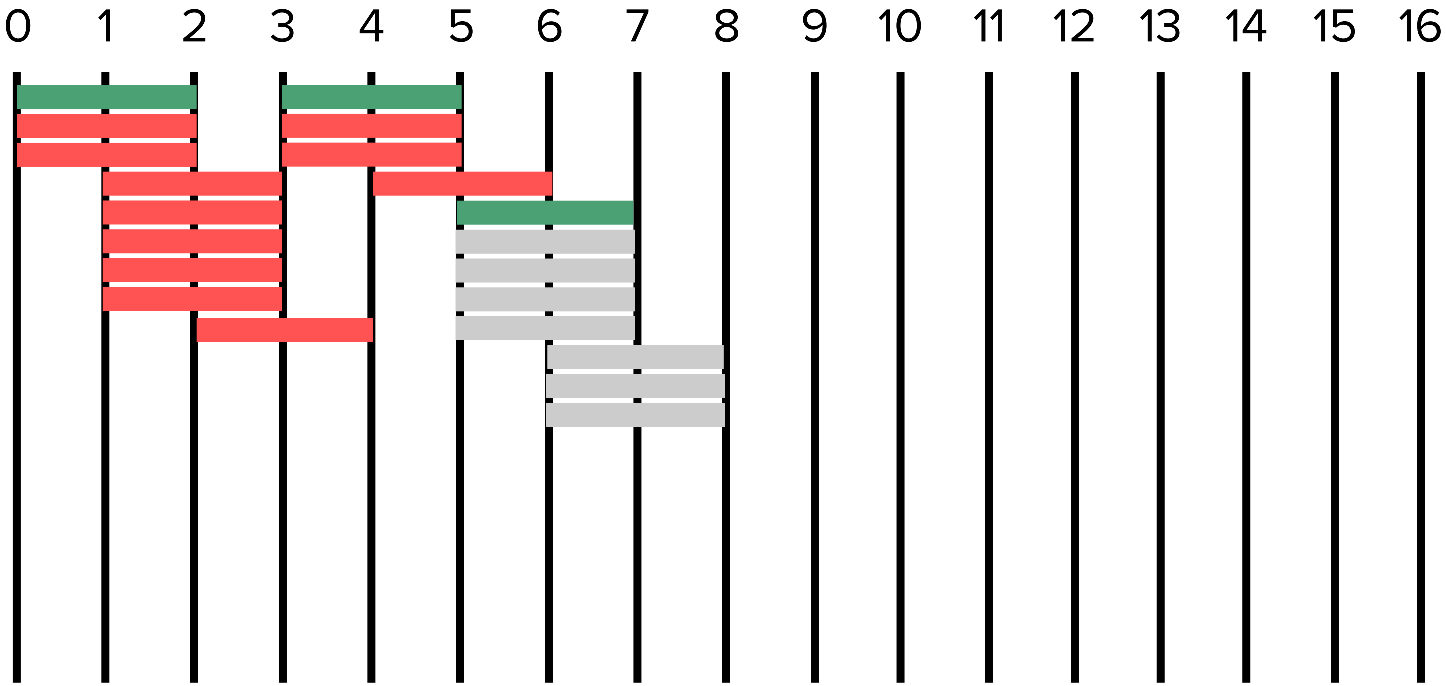

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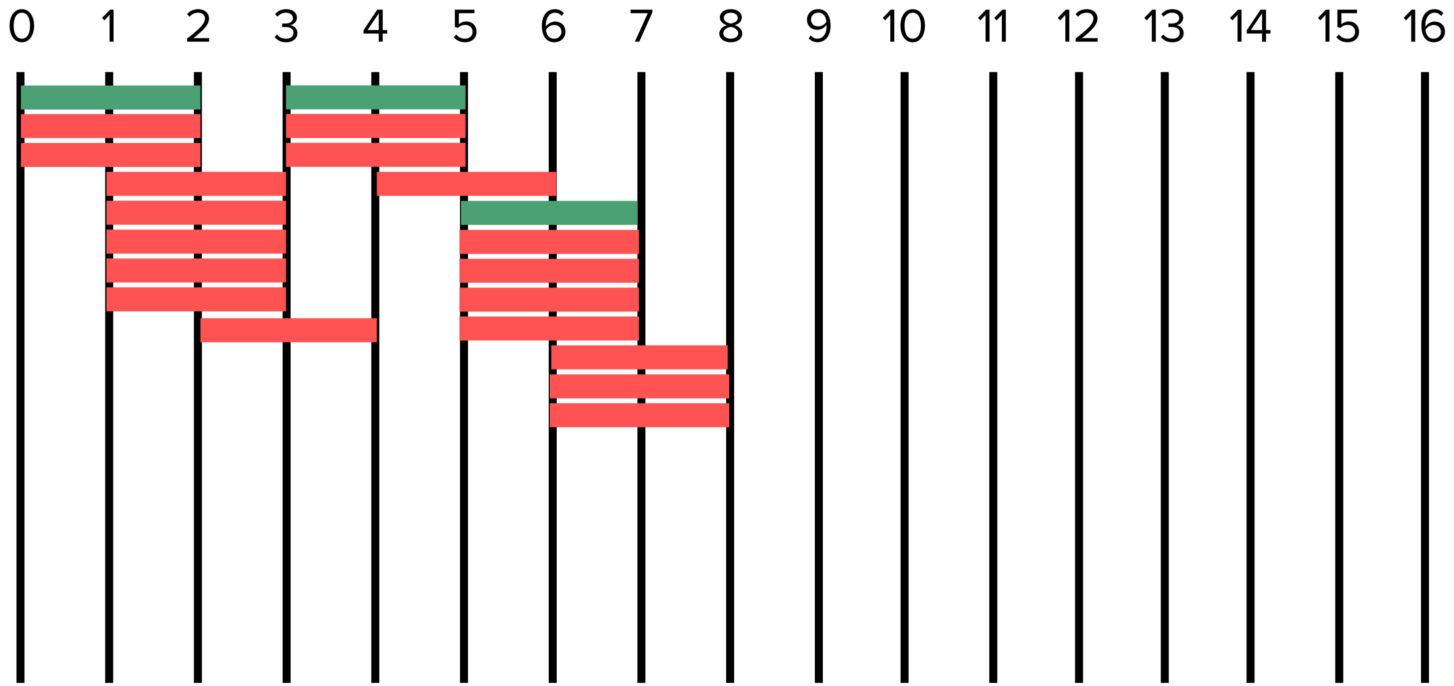

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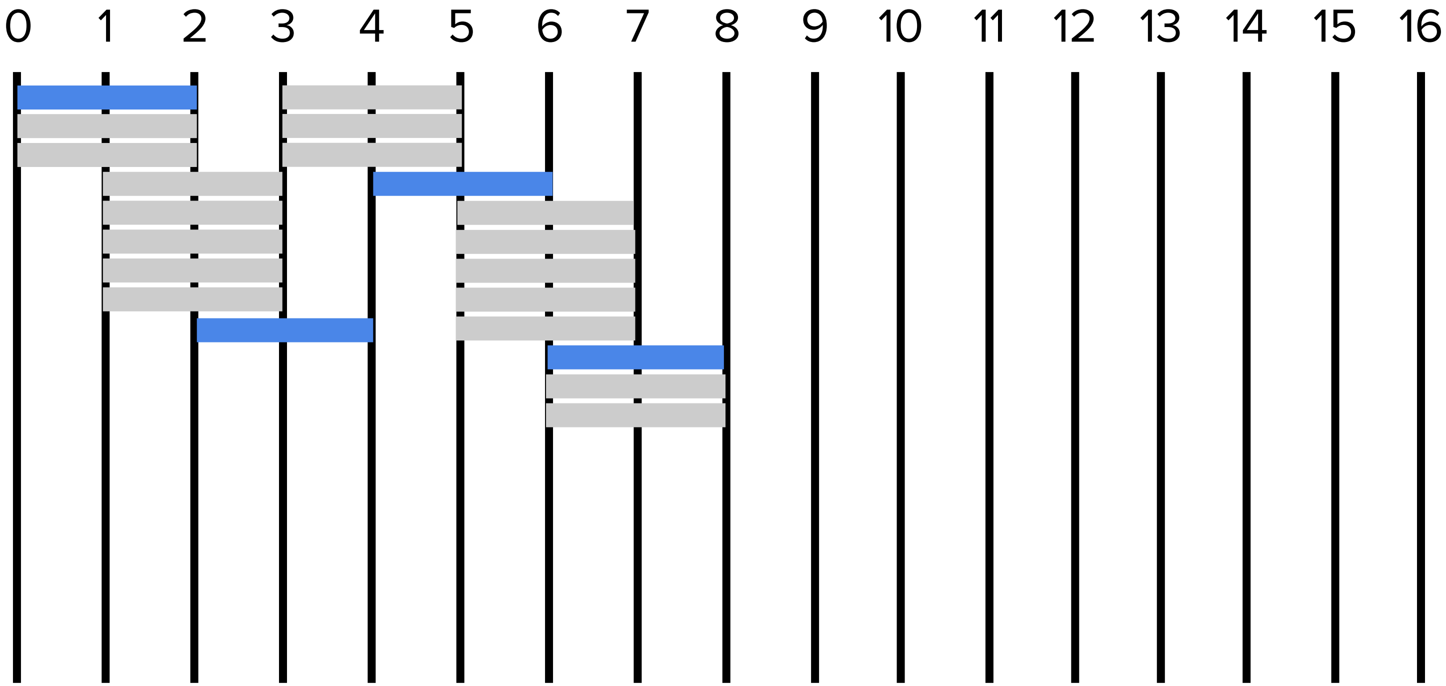

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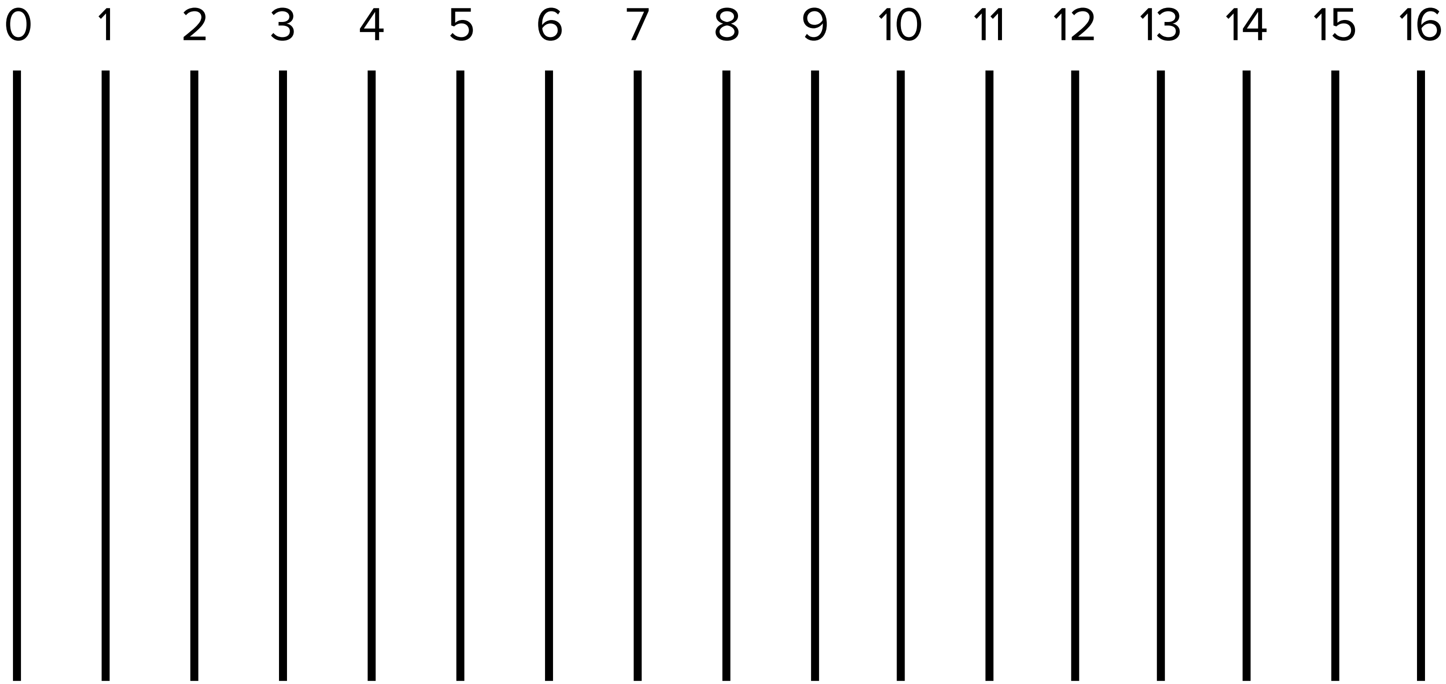

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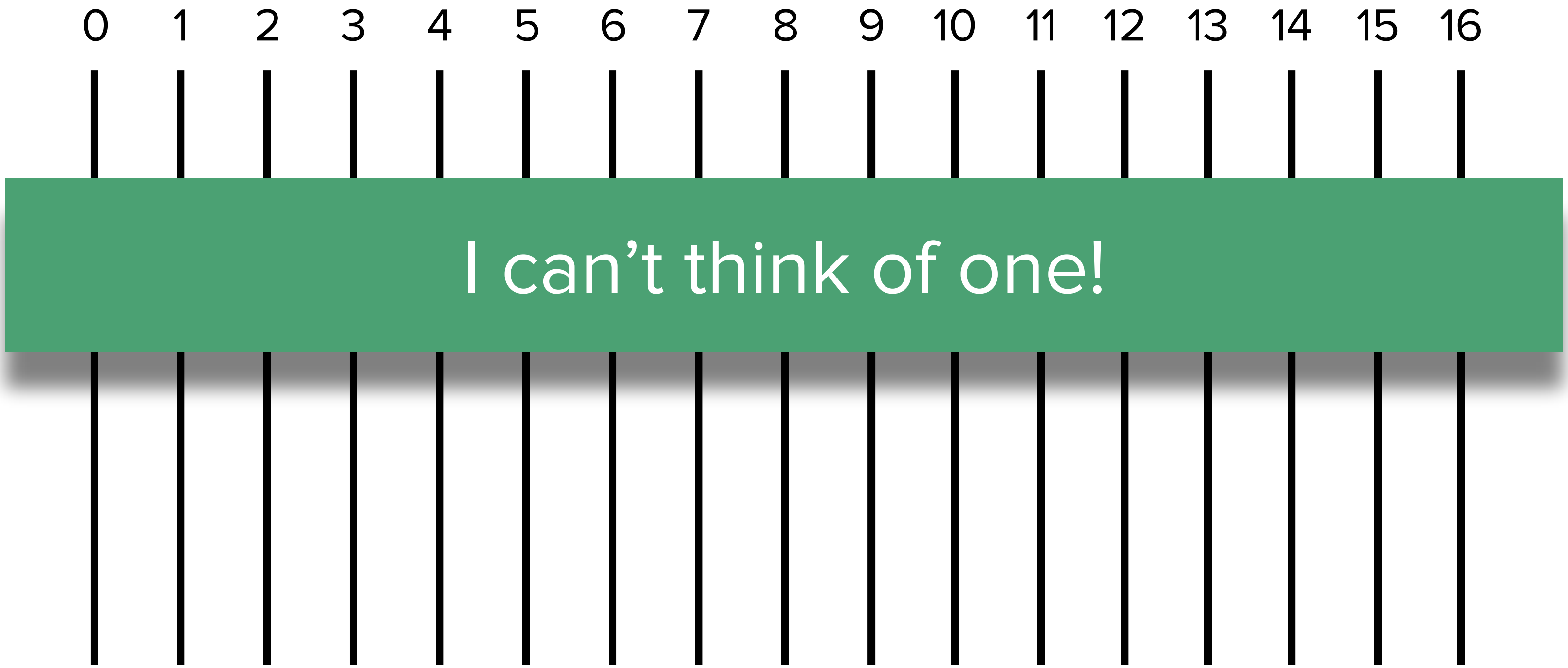

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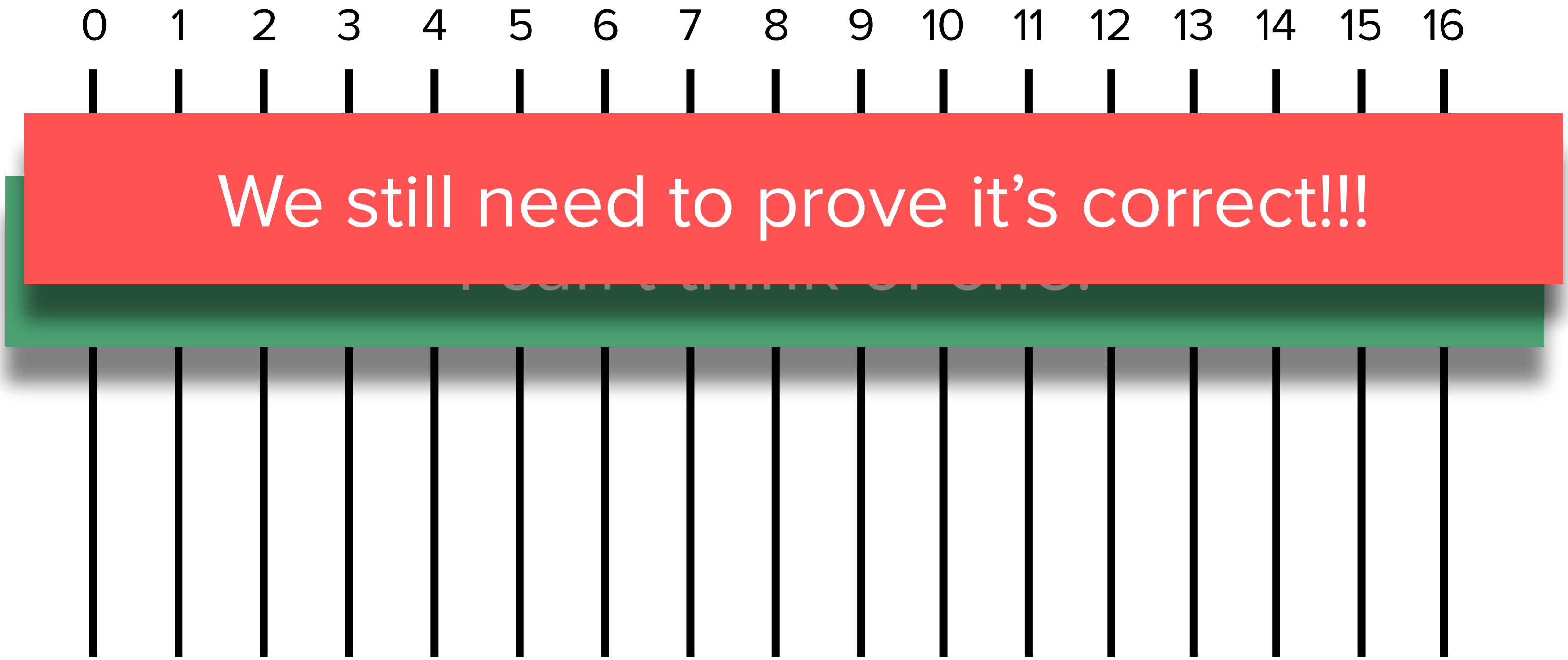

We still need to prove it's correct!!!

Example: The Event Scheduling Problem

```
Algorithm schedule(E):
    Sort E in ascending order of end time
    curr_time ← negative infinity
    events ← empty list
    For each event (start,end) in E:
        If start ≥ curr_time:
            Add (start,end) to events
            curr_time ← end
    Return events
```

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Let's try to prove our algorithm!

  - Create *S*' by exchanging a choice in *S* with *g* and show that
    - *S*' is a valid solution
    - *S*' is just as good, or better than, *S*